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## **MATHEMATICAL MODEL OF CORIOLIS VIBRATORY GYROSCOPES MOTION TRAJECTORY**

### **Introduction**

In relatively recent times Coriolis vibratory gyroscopes (CVGs) received significant amount of interest from the engineering community due to the promising possibility to fabricate sensitive elements of such gyroscopes in miniature form by using modern microelectronic mass-production technologies. At the same time, in comparison to other types of angular rate sensors, such gyroscopes provide designers with rich on feature measuring capabilities, including possibility to implement integrating gyro [1]. However, this often requires quite sophisticated analysis of the sensitive element motion, in order to make possible building different types of service control systems targeting different aspects of CVG operation. One of the problems, associated with the CVG control systems development, has been recently resolved, when simple and yet accurate transfer function of the sensitive element was derived [2]. Another set of problems, such as quadrature error compensation and trajectory control, requires analysis of the sensitive element trajectory, which has not yet been addressed due to the peculiarities of the CVG mathematical models.

In this paper we are going to address this problem and derive relatively simple mathematical model of the sensitive element trajectory parameters.

### **Problem formulation**

One could consider the CVG sensitive element as a two-dimensional pendulum, whose steady state trajectory forms a rotated ellipse, as shown in Figure 1. In this figure,  $a$  and  $b$  are the big and small half-axes of the ellipse,  $\theta$  is the angle of the ellipse rotation relatively to the axes of primary  $x_1$  and secondary  $x_2$  oscillations. It is well-known, that these parameters (namely half-axes and angle of rotation) depend on amplitudes and phases of primary and secondary oscillations, which in turn depend on parameters of the sensitive element design and unknown angular rate.

The problem, which is to be addressed in this paper, is to develop and analyse mathematical model of the ellipse parameters as a functions of the sensitive element design and its rotation.

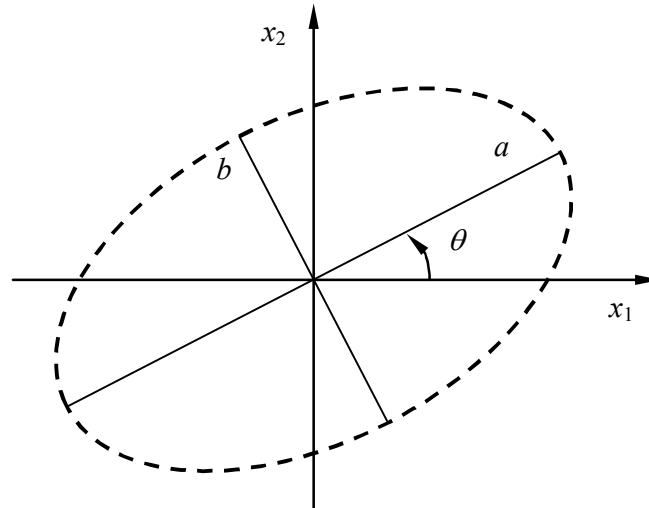


Fig. 1. Sensitive element motion trajectory

### Ellipse parameters

Without loss of generality, primary and secondary coordinates of the CVG sensitive element in its steady motion can be represented as

$$x_1(t) = A_1 \cos \omega t, \quad (1)$$

$$x_2(t) = A_2 \cos(\omega t + \varphi) = A_2 [\cos \omega t \cos \varphi - \sin \omega t \sin \varphi],$$

where  $A_1$  and  $A_2$  are the primary and secondary amplitudes,  $\varphi$  is the phase shift between the primary and secondary oscillations,  $\omega$  is the circular oscillations frequency. First of all, we have to exclude terms containing oscillation phase  $\omega t$  from the equations (1). For that we express sine and cosine of the  $\omega t$  from the first equation and substitute them into the second one, yielding

$$\frac{x_1}{A_1} \cos \varphi - \frac{x_2}{A_2} = \pm \sqrt{1 - \frac{x_1^2}{A_1^2}} \sin \varphi. \quad (2)$$

Squaring both sides of the equation (2) results in

$$\frac{x_1^2}{A_1^2} + \frac{x_2^2}{A_2^2} - \frac{2x_1x_2 \cos \varphi}{A_1A_2} = \sin^2 \varphi. \quad (3)$$

Assuming that  $X_1$  and  $X_2$  are the coordinates of the sensitive element in the coordinate system that is rotated by the angle  $\theta$ , they are related to the original coordinates as

$$\begin{aligned} x_1 &= X_1 \cos \theta - X_2 \sin \theta, \\ x_2 &= X_1 \sin \theta + X_2 \cos \theta. \end{aligned} \quad (4)$$

In these coordinates ellipse equation must have its canonical form

$$\frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} = 1.$$

Substituting (4) into (3) gives

$$\begin{aligned}
 & X_1^2 \frac{A_2^2 \cos^2 \theta - A_1 A_2 \cos \varphi \sin 2\theta + A_1^2 \sin^2 \theta}{A_1^2 A_2^2} + \\
 & + X_1 X_2 \frac{(A_1^2 - A_2^2) \sin 2\theta - 2A_1 A_2 \cos \varphi \cos 2\theta}{A_1^2 A_2^2} + \\
 & + X_2^2 \frac{A_1^2 \cos^2 \theta + A_1 A_2 \cos \varphi \sin 2\theta + A_2^2 \sin^2 \theta}{A_1^2 A_2^2} = \sin^2 \varphi.
 \end{aligned} \tag{5}$$

Apparently, second term in the equation (5) must disappear to make this equation canonical. This will occur if  $\theta$  satisfies the following condition:

$$\theta = \frac{1}{2} \arctan \frac{2A_1 A_2 \cos \varphi}{A_1^2 - A_2^2}. \tag{6}$$

If angle  $\theta$  is taken according to (6) then half-axes of the ellipse will be given by the following expressions:

$$\begin{aligned}
 a &= \frac{A_1 A_2 \sin \varphi}{\sqrt{A_2^2 \cos^2 \theta - A_1 A_2 \cos \varphi \sin 2\theta + A_1^2 \sin^2 \theta}}, \\
 b &= \frac{A_1 A_2 \sin \varphi}{\sqrt{A_1^2 \cos^2 \theta + A_1 A_2 \cos \varphi \sin 2\theta + A_2^2 \sin^2 \theta}}.
 \end{aligned} \tag{7}$$

Now we have to express half-axes  $a$  and  $b$  of the ellipse, and its angle of rotation  $\theta$  in terms of the sensitive element parameters and external angular rate.

### **Zero phase shift approximation**

In perfectly tuned CVG, which means that primary and secondary eigenfrequencies are matched, phase shift is zero ( $\varphi = 0$ ). This allows us to approximate expressions (6) and (7) with the linear terms of its Taylor series representation around zero phase shift:

$$\begin{aligned}
 a &\approx \frac{A_1 A_2 \varphi}{A_2 \cos \theta - A_1 \sin \theta}, \\
 b &\approx \frac{A_1 A_2 \varphi}{A_1 \cos \theta + A_2 \sin \theta}.
 \end{aligned} \tag{8}$$

One should note, that in approximations (8) zero phase shift results in zero big half-axis  $a$ , which apparently is not acceptable in trajectory analysis applications.

Let us study in greater details dependencies (7) for the elliptical trajectory of the CVG sensitive element. Obviously, formula (7) for the big half-axis has a peculiarity in vicinity of the zero phase shift, when denominator of the formula may become zero. However, far from the zero phase, this function is quite smooth and has no such peculiarities. From the extensive analysis of this dependency the following simple approximation is suggested:

$$a \approx A_1 + \frac{A_2^2}{4}(1 + \cos 2\varphi). \quad (9)$$

Relative error of the approximation (8) is shown in Figure 2.

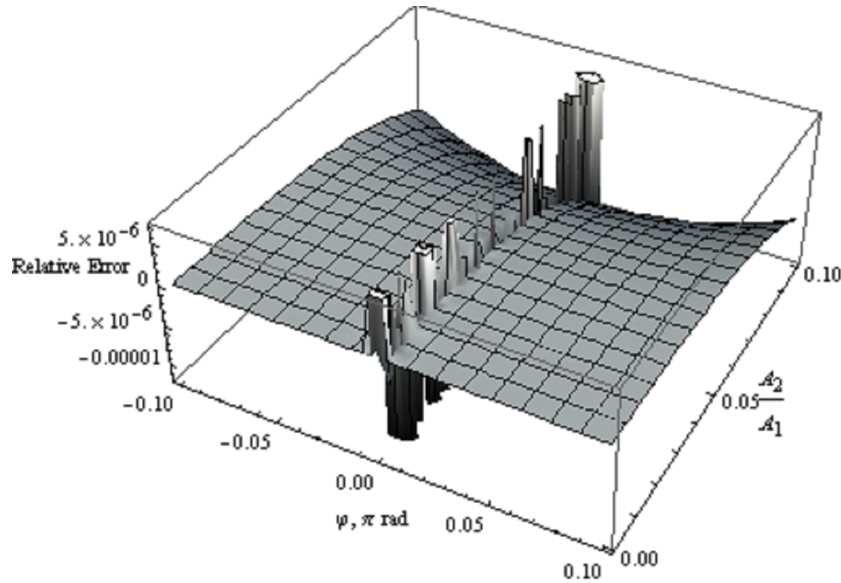


Fig. 2. Relative error of the big half axis approximation

Observable singularities along the zero phase shift in Figure 2 demonstrate that approximation is more relevant than the initial formula. It is easy to see, that approximation (9) is certainly accurate enough for small phase shifts and small relative secondary amplitudes, which are typical for the CVG sensitive element motions.

Let us now analyse how elliptical trajectory parameters depend on the external angular rate and sensitive element characteristics.

### Primary and secondary oscillations

In the most generalized form, motion equations of the CVG sensitive element both with translational and rotational motion could be represented in the following form [3]:

$$\begin{cases} \ddot{x}_1 + 2\zeta_1 k_1 \dot{x}_1 + (k_1^2 - d_1 \Omega^2)x_1 + g_1 \Omega \dot{x}_2 + d_3 \dot{\Omega} x_2 = q_1(t), \\ \ddot{x}_2 + 2\zeta_2 k_2 \dot{x}_2 + (k_2^2 - d_2 \Omega^2)x_2 - g_2 \Omega \dot{x}_1 - \dot{\Omega} x_1 = q_2(t). \end{cases} \quad (10)$$

Here  $x_1$  and  $x_2$  are the generalized coordinates that describe primary (excited) and secondary (sensed) motions of the sensitive element respectively,  $k_1$  and  $k_2$  are the corresponding natural frequencies,  $\zeta_1$  and  $\zeta_2$  are the dimensionless relative damping coefficients,  $\Omega$  is the measured angular rate, which is orthogonal to the axes of primary and secondary motions,  $q_1$  and  $q_2$  are the generalized accelerations due to the external forces acting on the sensitive element. The remaining dimensionless coefficients are different for the sensitive

elements exploiting either translational or rotational motion. For the translational sensitive element they are  $d_1 = d_2 = 1$ ,  $d_3 = m_2/(m_1 + m_2)$ ,  $g_1 = 2m_2/(m_1 + m_2)$ ,  $g_2 = 2$ , where were  $m_1$  and  $m_2$  are the masses of the outer frame and the internal massive element. In case of the rotational motion of the sensitive element, these coefficients are the functions of different moments of inertia (for greater details see [3]).

Steady state solution of the equations (10) in terms of amplitudes and phases of primary and secondary oscillations can be represented as follows:

$$\begin{aligned}
 A_1 &= \frac{q_{10}}{k^2 \sqrt{(1 - \delta\omega^2)^2 + 4\zeta_1^2 \delta\omega^2}}, \\
 A_2 &= \frac{A_1 g_2 \delta\omega}{\sqrt{\delta k^4 + \delta\omega^2 - 2\delta k^2 \delta\omega^2 (1 - 2\zeta_2^2)}} \delta\Omega, \\
 \varphi_1 &= -\arctan \frac{2\delta\omega\zeta_1}{1 - \delta\omega^2}, \\
 \varphi_2 &= -\arctan \frac{\delta k^2 - (1 + 4\zeta_1\zeta_2 \delta k + \delta k^2)\delta\omega^2 + \delta\omega^4}{2\delta k \delta\omega (\zeta_2 + \zeta_1 \delta k) - 2\delta\omega^3 (\zeta_1 + \zeta_2 \delta k)}.
 \end{aligned} \tag{11}$$

Here  $q_{10}$  is the amplitude of accelerations created by the primary excitation system,  $k$  is the primary natural frequency,  $\delta\omega = \omega/k$  is the relative excitation frequency,  $\delta k = k_2/k_1$  is the ratio of the secondary and primary natural frequencies,  $\delta\Omega = \Omega/k$  is the relative angular rate. Angular rate is assumed to be negligible in comparison to the natural frequencies.

One should note that the sensitive element trajectory parameters depend on the phase shift  $\varphi = \varphi_2 - \varphi_1$  between primary and secondary phases. Most importantly, based upon (9), phases of do not depend on angular rate.

In case of primary resonance ( $\delta\omega = 1$ ), sine and cosine of this phase shift can be calculated as

$$\begin{aligned}
 \sin \varphi &= \frac{1 - \delta k^2}{R}, \\
 \cos \varphi &= \frac{2\zeta_2 \delta k}{R}, \\
 R &= \sqrt{\delta k^4 - 2(1 - 2\zeta_2^2)\delta k^2 + 1}.
 \end{aligned} \tag{12}$$

Expressions (11) along with the phase shift representations (12) can now be used to analyse parameters of the actual trajectory of the CVG sensitive element.

### **Sensitive element motion trajectory parameters**

Let us now substitute expressions (11) and (12) into the formula for the big half-axis approximation (9):

$$a = \frac{q_{10}(4k^2R^2\zeta_1 + g_2^2q_{10}\delta\Omega^2)}{8k^4R^2\zeta_1^2}. \quad (13)$$

In case of the small angular rates ( $\delta\Omega^2 \approx 0$ ), big half-axis, as expected, becomes the primary amplitude:

$$a \approx \frac{q_{10}}{2k^2\zeta_1} = A_1|_{\delta\Omega=0}.$$

Small half axis of the sensitive element trajectory is given as

$$b = \frac{q_{10}g_2(1 - \delta k^2)}{\sqrt{2k^2R\zeta_1}\sqrt{R^2 + g_2^2\delta\Omega^2 + \sqrt{R^4 - 2g_2^2(R^2 - 8\zeta_2^2\delta k^2)\delta\Omega^2 + g_2^4\delta\Omega^4}}}\delta\Omega. \quad (14)$$

Both big and small half-axes, as a functions of the angular rate and natural frequency ratio are shown in Figure 3.

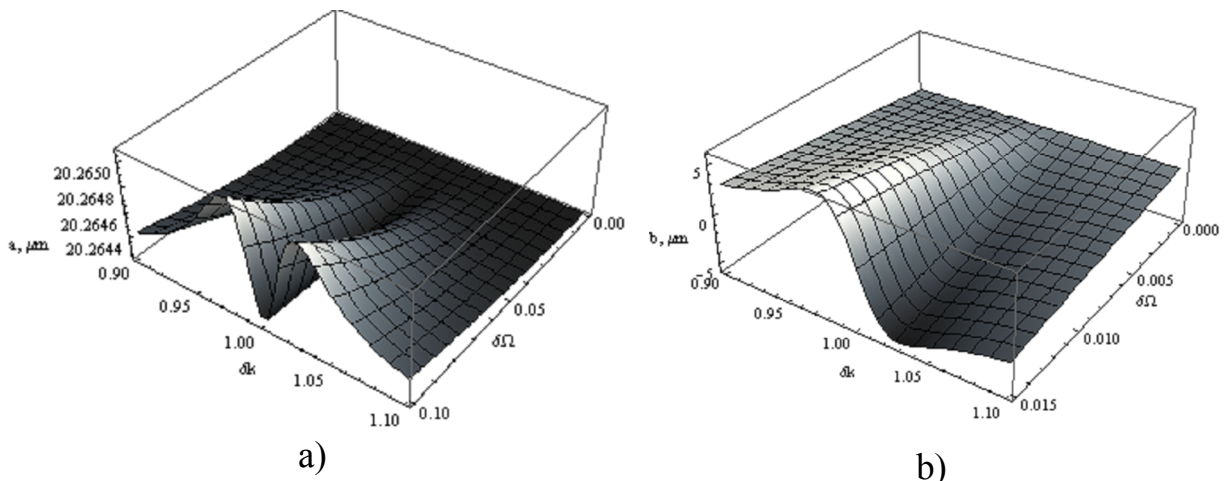


Fig. 3. Half axes of the sensitive element trajectory as a functions of the angular rate and natural frequency ratio (a – big half axis, b – small half axis)

Formula (13) can be accurately replaced with its linear approximation yielding:

$$b \approx \frac{q_{10}g_2(1 - \delta k^2)}{2k^2\zeta_1[1 - 2(1 - 2\zeta_2^2)\delta k^2 + \delta k^4]}\delta\Omega. \quad (15)$$

Analysing formulae (14) or (15) one can see, that small half axis is absent in either of two cases: perfect match of the natural frequencies ( $\delta k = 1$ ) or absence of the external angular rate ( $\delta\Omega = 0$ ).

Finally, the last but not the least, angle of the trajectory rotation  $\theta$  can be calculated using the following expression:

$$\theta = \frac{1}{2} \arctan \left[ \frac{4g_2\zeta_2\delta k\delta\Omega}{1 - 2(1 - 2\zeta_2^2)\delta k^2 + \delta k^4 - g_2^2\delta\Omega^2} \right]. \quad (16)$$

Similarly to the previous case, we can linearly approximate formula (16) for small angular rates:

$$\theta = \frac{2g_2\zeta_2\delta k\delta\Omega}{1 - 2(1 - 2\zeta_2^2)\delta k^2 + \delta k^4}. \quad (17)$$

Or, in case of perfectly matched primary and secondary natural frequencies ( $\delta k = 1$ ), it becomes

$$\theta \approx \frac{g_2}{2\zeta_2} \delta\Omega. \quad (18)$$

Both accurate expression (16) and its linear approximation (18) are shown in the figure 4 below.

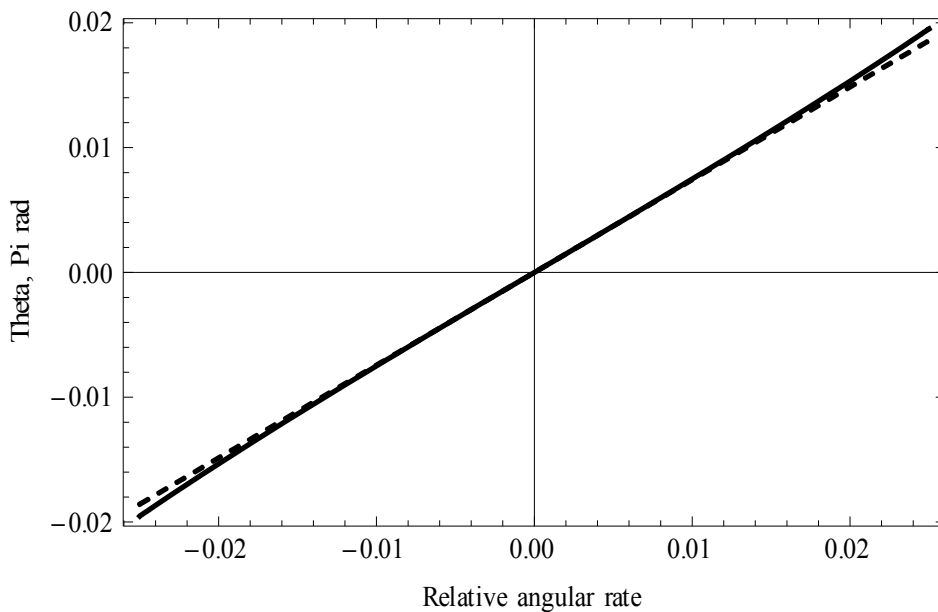


Fig. 4. Trajectory rotation angle as a function of angular rate (solid – accurate, dashed – linear approximation)

From the graphs in Figure 4 one could see, that trajectory rotation angle is almost linear function of the unknown angular rate, which makes it possible to use this parameter to measure angular rate. Let us finally verify obtained dependencies for the CVG sensitive element motion trajectory parameters by means of comparison with the direct numerical simulations based on the equations (10). Resulting trajectories are presented in Figure 5 below.

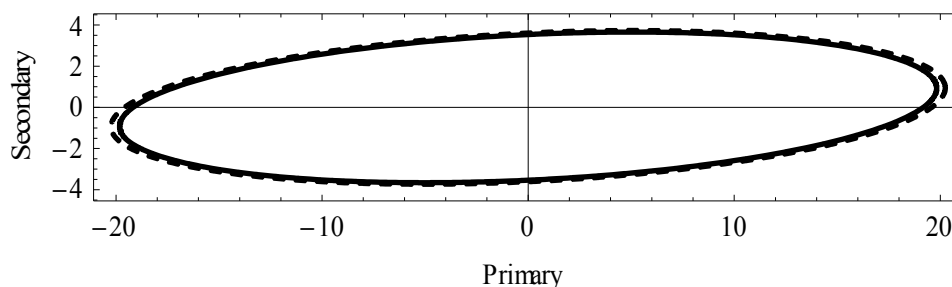


Fig. 5. CVG Trajectory simulation (solid – simulated, dashed – theoretical)

Here solid line demonstrates steady motion trajectory of the simulated CVG sensitive element, and dashed line corresponds to the trajectory, generated from the obtained in the paper ellipse parameters.

### **Conclusions**

Both accurate and approximated new dependencies for the steady motion trajectory parameters, such as half axes and rotation angle, of the CVG sensitive element as functions of CVG design characteristics and external angular rate were obtained in this paper. Comparison with the results of numerical simulation demonstrates high accuracy of the obtained mathematical models. These dependencies allow not only further analysis of the sensitive element motion, but efficient synthesis of many different control loops improving overall CVG performance as well. The latter suggested as topics of the future research.

### **References**

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